HOMEWORK 6 – SAMPLE SOLUTION

1. Exercise 26.3-4, page 669

necessary part: A perfect matching is a matching in which every vertex is matched. Thus, if there is a perfect matching for a graph G, then every vertex $v \in L$ has at least a neighbor $u \in R$. It implies that $|A| \leq |N(A)|$ for every subset $A \subseteq L$

sufficient part: We could prove it by induction on the number of vertices of L.

In base case, this theorem is trivially true for $|L| = 0$.

Now we assume the theorem true for all $|L| < n$, we prove it for $|L| = n$.

First suppose that we have the stronger condition $|A| < |N(A)|$ for all $A \subset L$. We could pick any $v \in L$, and one of its neighbor $u \in R$. Discard these two vertices in L and R. We will have $|A| \leq |N(A)|$ for all $A \subseteq L - \{v\}$. Thus, we could prove this case by the already-proven case of the theorem.

Otherwise, there should exist a subset $T \subset L$ such that $|N(T)| = |T|$. Inside T, for all $A \subseteq T \subset L$, we still have $|A| \leq |N(A)|$, so by an already-proven case of the theorem. There is a perfect matching between T and $N(T)$.

We still need to prove that $L - T$ could perfect match to $R - N(T)$. It sufficient to show that for any subset $A \subseteq L - T$, we have $|A| \leq |N(A) - N(T)|$.

$$|N(A)-N(T)| = |N(A)\cup N(T)|-|N(T)| = |N(A\cup T)|-|N(T)| \geq |A\cup T|-|T| = |A|+|T|-|T| = |A| \tag{0.0.1}$$

2. Show how to model an extension of the network flow problem if, in addition to capacities on the edges, there are also positive capacities on the vertices: the sum of the flows into a vertex, v, (other than s, t) must be $\leq$ capacity(v). That is, given such a problem, find an equivalent network flow problem as specified in the text.

The capacity of flow was on the edges for network flow problem. Thus, for each vertex v of G, we would introduce it as two vertices v_0, v_1 with the capacity of vertex v on the edge (v_0, v_1). For any incoming edges to v in G, we have them as incoming edges to v_0 in new graph, and for any outgoing edges from v in G, we have them as outgoing edges from v_1. The new graph with capacities on edges only satisfies all of the properties of flow networks.

3. Exercise 34.4-4, page 1002

Tautology is essentially the same as the complement of SAT problem.

And we have known that SAT is a NP-complete problem. So we will show that L is NP-complete if and only if its complement $\overline{L}$ is co-NP-complete.

Let $\overline{L} \in$ co-NP. Hence $L \in$ NP. If L is NP-complete ,then there is a polynomial-time reduction function R from L to SAT such that for any input x we have $x \in L$ if and only if $R(x)$ is satisfiable. Clearly, for any x not in L if and only if $R(x)$ is not satisfiable. It implies that for any $x \in \overline{L}$ if and only if $R(x)$ is not satisfiable. It means that $\overline{L} \in$ co-NP.

Received by the editors Jan., 2008.

4. Show that the following problem is NP-complete. Given a list of positive integers, L, can the list be partitioned into three lists, $L = L_1 \cup L_2 \cup L_3, L_i \cap L_j = \emptyset$ for $i \neq j$, such that the sums of the integers in each list are equal. You may assume that the subset sum problem is NP-complete

Denote this problem as 3-EQUAL-SET. We first show this problem $\in$ NP. Suppose we are given a list of positive integers L. Let L_1, L_2, L_3 be the certificate where $L = L_1 \cup L_2 \cup L_3, L_i \cap L_j = \emptyset$ for $i \neq j$. We would only need to check whether $\sum(L_1) = \sum(L_2) = \sum(L_3)$. It could be done in polynomial time.

We prove that 3-EQUAL-SET is NP-hard by showing that SUBSET-SUM $\leq_p$ 3-EQUAL-SET. Given a set S of n positive integers, and an integer k. We define $L = S \cup a_{n+1} \cup a_{n+2}$, where $a_{n+1} = k, a_{n+2} = 2k - \sum(S)$. Without lose of generality, we could assume $2k - \sum(S) \geq 0$, since for any SUBSET-SUM problem (S, k) is always equal to another SUBSET-SUM problem (S, $\sum(S) - k$). So we could always assume $2k \geq \sum(S)$. Clearly, the reduction from subset sum problem to this problem could be done in polynomial time. To complete the proof. We will show that the set S has a subset sum k if and only if L could be partitioned into three subsets such that the sum of each subset is the same.

Suppose that S has a subset S' of sum k, then $L = S \cup a_{n+1} \cup a_{n+2}$ could be partitioned into three subset as $S', \{a_{n+1}\}, S - S' \cup a_{n+2}$. Since $\sum(S') = k, a_{n+1} = k$, and $S - S' \cup a_{n+2} = \sum(S) - k + 2k - \sum(S) = k$. Clearly, we get a feasible solution to L.

Conversely, if there is no feasible solution for the set S with integer k, then we can never partition L into three subsets with the sum of each subset is the same as other subset, since $\sum(L) = 3 \times k$, the sum of each subset should be equal to k. And $a_{n+1} = k$ should be a good subset, since it has only one integer. However there is no subset of S such that of the sum is equal to k. So the set $S \cup a_{n+2}$ could not be partitioned into two sets of the sum are equal to k.

Then we could conclude that 3-EQUAL-SET is NP-complete.

5. Show that the following problem is NP-complete. Given a directed graph, $G = (V, E)$ does G have two different Hamiltonian circuits?

Denote this problem as 2-DHC. We first show this problem $\in$ NP. Suppose we are given a graph $G = (V, E)$. And $E_1, E_2 \in E$ as a certificate. We would check whether E_1 and E_2 form Hamiltonian circuits by traversing all of the edges in E_1 and E_2. We also could check $E_1 \cap E_2 = \emptyset$. It could be done in polynomial time.

We prove that 2-DHC is NP-hard by showing that HAM-CYCLE $\leq_p$ 2-DHC. Given a graph $G = (V, E)$. We define $G' = (V', E')$, where for each vertex $v \in V$, we have a vertex $v' \in V'$, and for each edge $e = (u, v) \in E$, we introduce two new directed edges $e = (u', v'), e' = (v', u') \in E'$

Clearly, this reduction could be done in polynomial time. To complete the proof. We will show that the graph G has a Hamiltonian cycle if and only if G' has two directed Hamiltonian cycle.

Suppose that G has a Hamiltonian cycle. There should be an ordered set of edges $\{e_0, e_1, \ldots, e_{n-1}\}$ that we traverse from a start vertex and visit all of the vertices and go back to itself. Thus there also should be a solution of two hamiltonian cycles: one is $\{e_0, e_1, \ldots, e_{n-1}\}$, and another one is $\{e'_{n-1}, e'_{n-2}, \ldots, e'_0\}$ in G'.

Conversely, if there is a solution for G', it would has two hamiltonian cycles for G'. Since every edge e in G' has one direction should has another edge e' in G' has another direction. Thus, it implies there should be a Hamiltonian cycle in G.

Then we could conclude that 2-DHC is NP-complete.

6. Exercise 34.5-1, page 1017

We are given two graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$, and asks whether G_1 is isomorphic to a subgraph of G_2. We first show this problem $\in$ NP. Given a certificate $f : V_1 \rightarrow V_2$. We could check whether every edge in $(u, v) \in E_1$ has an edge $(f(u), f(v)) in E_2$ in polynomial time.

Second, we prove that this problem is NP-hard by showing that Clique $\leq_p$ Subgraph-Isomorphism. For any Clique problem (G, k), we construct a k-clique graph G_1, and a graph G_2 that is identical the same to G for Subgraph-Isomorphism problem.

Clearly, if there is a solution for Clique (G, k) problem, it implies that there exists a k-clique in G. Thus, G_1 should be isomorphic to a subgraph of G_2, since G_2 is the same as G, G has a k-clique, and G_1 is a k-clique.

Conversely, if there is no feasible solution for Clique (G, k) problem, there should be no way to find a k-clique in G_2, so G_1 is not isomorphic to any subgraph of G_2

Then we could conclude that Subgraph-Isomorphism is NP-complete.

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